ON COMPLEX BANACH MANIFOLDS SIMILAR TO STEIN MANIFOLDS

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ABSTRACT. We give an abstract definition, similar to the axioms of a Stein manifold, of a class of complex Banach manifolds in such a way that a manifold belongs to the class if and only if it is biholomorphic to a closed split complex Banach submanifold of a separable Banach space.

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Stein manifolds can be characterized among complex manifolds in various ways, including the two ways (I) and (II) below. A paracompact second countable Hausdorff complex manifold M of pure dimension is a Stein manifold if and only if one and hence both of the following equivalent conditions (I) and (II) below hold.

(I) (a) M is holomorphically convex, i.e., if $K \subset M$ is compact, then its $\mathcal{O}(M)$ holomorphic hull $\hat{K}$ is compact in M. (b) If $x \neq y$ in M, then there is an $f \in \mathcal{O}(M)$ with $f(x) \neq f(y)$. (c) If $x \in M$, then there are an integer $n \geq 1$ and a holomorphic function $g \in \mathcal{O}(M, \mathbb{C}^n)$ that is a biholomorphism from an open neighborhood W of x in M to an open neighborhood $g(W)$ of $g(x)$ in $\mathbb{C}^n$.

(II) There is an $n \geq 1$ such that M is biholomorphic to a closed complex submanifold M' of $\mathbb{C}^n$.

Let X be a separable Banach space, and M a paracompact second countable Hausdorff complex Banach manifold modelled on X. We call M a *Stein Banach manifold modelled on* X if (i-iv) below hold.

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(i) Holomorphic domination is possible in M, i.e., if $u\colon M \to \mathbb{R}$ is any locally upper bounded function, then there are a Banach space Z and a holomorphic function $h\colon M \to Z$ with $u(x) < \|h(x)\|$ for all $x \in M$.

(ii) There are open sets $U_n, V_n \subset M$, and holomorphic functions $f_n \in \mathcal{O}(M)$, $n \geq 1$, such that $\bigcup_{n=1}^{\infty}(U_n \times V_n) = (M \times M) \setminus \Delta_M$, where $\Delta_M = \{(x,x)\colon x \in M\}$ is the diagonal of $M \times M$, and $f_n(U_n)$ and $f_n(V_n)$ are disjoint sets in $\mathbb{C}$ for all $n \geq 1$.

(iii) There are open sets $W_n \subset M$ and holomorphic maps $g_n \in \mathcal{O}(M, X)$ for $n \geq 1$ such that $\bigcup_{n=1}^{\infty} W_n = M$ and $g_n|W_n$ is a biholomorphism from W_n onto an open set $g_n(W_n)$ in X.

(iv) There are open sets $G_k \subset M$, $k \geq 1$, with $\bigcup_{k=1}^{\infty} G_k = M$ and $\sup_{x \in G_k}(|f_n(x)| + \|g_n(x)\|) < \infty$ for all $k, n \geq 1$, where (f_n) and (g_n) are as in (ii) and (iii).

If M is finite dimensional, then it is easy to see that (i-iii) together are equivalent to (I), and (iv) is vacuous, since if G_k, $k \geq 1$, is an exhaustion of M by precompact open sets G_k, then any continuous function $|f_n(x)| + \|g_n(x)\|$ on M is bounded on G_k for $k, n \geq 1$. Thus if M is finite dimensional, then (i-iv) together are equivalent to M being a Stein manifold.

Theorem 1. *Let X be a separable Banach space, and M a paracompact second countable Hausdorff complex Banach manifold modelled on X. Then M is a Stein Banach manifold modelled on X if and only if there is a separable Banach space X' such that M is biholomorphic to a closed split complex Banach submanifold M' of X'.*

Here Banach manifolds and Banach submanifolds are understood in terms of biholomorphically related charts, and a Banach submanifold is called *split* if each of its tangent spaces has a direct complement in the ambient Banach space. Clearly, a complex Banach submanifold M of X is split if and only if near each point $x_0 \in M$ it is possible to split X as a direct sum $X = X' \times X''$ of closed linear subspaces X', X'' of X such that with $x_0 = (x_0', x_0'')$ and $x = (x', x'')$ we can write M as the graph $x'' = m(x')$ of a holomorphic function m from an open neighborhood of x_0' in X' to X'', where $x_0'' = m(x_0')$.

Proof. Suppose first that M is biholomorphic to an M' and verify that M satisfies (i-iv). It is enough to show that M' does.

As holomorphic domination is possible in X' by [P], and thus also in M', since M' is closed in X', (i) is true. We define some linear functions $f_n\colon X' \to \mathbb{C}$ and $g_n\colon X' \to X$ for $n \geq 1$ whose restrictions to M' will do the job. For linear functions (iv) is automatic: we can let G_k be the intersection of M' with the open ball $\|x\| < k$ in X' and write $|f_n(x)| + \|g_n(x)\| \leq$

$(\|f_n\| + \|g_n\|)\|x\| \le (\|f_n\| + \|g_n\|)k < \infty$ for $x \in G_k$ and $n, k \ge 1$.

If $x \ne y$ in M', then $x - y \ne 0$ in X' and the Hahn–Banach theorem gives us a complex linear functional $f_{xy} \in (X')^*$ of norm 1 with $\operatorname{Re} f_{xy}(x-y) = \|x-y\| > 0$. Let $U_{xy} = \{z \in X'\colon -\frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(x) < \operatorname{Re} f_{xy}(z)\}$ and $V_{xy} = \{z \in X'\colon \operatorname{Re} f_{xy}(z) < \frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(y)\}$. Then $x \in U_{xy}$, $y \in V_{xy}$, and their images $f_{xy}(U_{xy})$ and $f_{xy}(V_{xy})$ are disjoint since they are the half planes $-\frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(x) < \operatorname{Re} w$, $\operatorname{Re} w < \frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(y)$, which are clearly disjoint since $-\frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(x) = \frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(y)$.

Fix any point $x_0 \in M'$ and denote its complex tangent space $T_{x_0}M'$ by X and regard it as a closed linear subspace of X'. If $x \in M'$, then the complex tangent space T_xM' and X are linearly isomorphic via a bounded linear map $i_x\colon T_xM' \to X$, and there is a bounded linear projection $p_x\colon X' = T_xX' \to T_xM'$. Thus the linear map $g_x\colon X' \to X$ given by $g_x(y) = i_x(p_x(y))$ for $y \in X'$ satisfies that $(dg_x)(x)y = i_x(y)$ for $y \in T_xM'$, i.e., $(dg_x)(x)$ is a linear isomorphism from T_xM' onto X. By the inverse function theorem g_x is biholomorphic from an open neighborhood W_x of x in M' to an open neighborhood $g_x(W_x)$ of $g_x(x) = 0$ in X.

By Lindelöf's theorem in the second countable (separable metric) spaces $(M' \times M') \setminus \Delta_{M'}$ and M' the open coverings $U_{xy} \times V_{xy}$, $(x, y) \in (M' \times M') \setminus \Delta_{M'}$, and W_x, $x \in M'$, can be reduced to countable subcoverings $U_n \times V_n$, W_n, where $U_n = U_{x_n y_n}$, $V_n = V_{x_n y_n}$, and $W_n = W_{x'_n}$ for $n \ge 1$. Thus the functions $f_n = f_{x_n y_n}$, $g_n = g_{x'_n}$, $n \ge 1$, do the job.

Conversely, assume that M satisfies (i-iv) and embed M biholomorphically as M' into a separable Banach space X'.

If $i \ge 1$, then let $C_i = L_i = 1 + \sup\{|f_n(x)| + \|g_n(x)\|\colon 1 \le k, n \le i, x \in G_k\}$. So if $k, n \ge 1$, and $x \in G_k$, then $|f_n(x)| + \|g_n(x)\| \le C_k L_n$. Thus upon replacing f_n by $f_n/(L_n 2^n)$ and g_n by $g_n/(L_n 2^n)$, we obtain new functions again to be called f_n, g_n that satisfy (ii), (iii), and the slightly strengthened version $\sup_{x \in G_k}(|f_n(x)| + \|g_n(x)\|) < C_k/2^n$, $k, n \ge 1$, of (iv).

The covering W_n, $n \ge 1$, of the paracompact space M has a locally finite refinement, which by Lindelöf's theorem can be taken to be countable, and can be shrunk since a paracompact Hausdorff space M is normal. There are open sets $M_n \subset M$, $n \ge 1$, with $\bigcup_{n=1}^\infty M_n = M$, and for each $n \ge 1$ there is an index $j(n) \ge 1$ with the closure $\overline{M_n} \subset W_{j(n)}$. Define $u\colon M \to \mathbb{R}$ by $u(x) = \inf\{n \ge 1\colon x \in M_n\}$. Then u is locally upper bounded on M since $u \le n$ on the open set M_n.

By assumption (i) on holomorphic domination there are a Banach space Z and a holomorphic function $h \in \mathcal{O}(M, Z)$ with $u(x) < \|h(x)\|$ for $x \in M$. As $Z' = \overline{\operatorname{span}}\{h(x)\colon x \in M\}$ is a separable Banach space, and as any separable

Banach space can be embedded into $C[0,1]$ we can replace the Banach space Z by the separable space $Z = C[0,1]$ endowed with the sup norm.

Define a Banach space X' by $X' = Z \times \ell_1 \times \ell_1(X)$, where ℓ_1 and $\ell_1(X)$ denote the spaces of summable sequences in $\mathbb{C}$ and in X. Let us write the variable y in X' as $y = (y', y''_n, y'''_n)$, where $y'' = (y''_n) \in \ell_1$ and $y''' = (y'''_n) \in \ell_1(X)$. Clearly, X' is a separable Banach space, being the product of three such spaces.

Define the map $\Phi\colon M \to X'$ given by $y = \Phi(x)$, where

$$\begin{cases} y' &= h(x) \\ y''_n &= f_n(x)\ , \qquad n \geq 1. \\ y'''_n &= g_n(x) \end{cases}$$

If $k \geq 1$ and $x \in G_k$, then we have $\sum_{n=1}^{\infty}(|f_n(x)| + \|g_n(x)\|) \leq \sum_{n=1}^{\infty} C_k/2^n \leq C_k < \infty$. Thus Φ is holomorphic.

Our Φ is injective, since if $x \neq y$ in M, then there is an index $n \geq 1$ with $x \in U_n$ and $y \in V_n$, so $f_n(x) \neq f_n(y)$, and even more so $\Phi(x) \neq \Phi(y)$.

We claim that the set $M' = \Phi(M)$ is closed in X'. Indeed, suppose that $y_i = \Phi(x_i)$, $x_i \in M$, converges $y_i \to y$ in the norm in X' as $i \to \infty$ to an element $y \in X'$, we must show that there is an $x \in M$ with $y = \Phi(x)$, i.e., $y \in M'$. As $y'_i = h(x_i)$, $i \geq 1$, is bounded, being convergent, there is an index $b \geq 1$, with $\|h(x_i)\| \leq b$ for $i \geq 1$, i.e., $u(x_i) < \|h(x_i)\| \leq b$, or, $x_i \in M_1 \cup \ldots \cup M_b$ for $i \geq 1$. By the pigeon hole principle there are an index a with $1 \leq a \leq b$ and an infinite set I of indices i such that $x_i \in M_a$ for all $i \in I$. As $\overline{M_a} \subset W_{j(a)}$, and $g_{j(a)}$ is biholomorphic on $W_{j(a)}$, we see that $\Phi(x_i)$, $i \in I$, may converge only if the x_i, $i \in I$, converge in $\overline{M_a}$ to one of its elements $x \in \overline{M_a} \subset W_{j(a)}$. Thus $y_i = \Phi(x_i) \to \Phi(x) = y$ as $i \to \infty$ in I. As M' contains y, it is closed.

If $y_0 = \Phi(x_0)$ in M', then there is an index $n \geq 1$ with $x_0 \in W_n$. So $y'''_n = g_n(x)$ is biholomorphic from a connected open set W'_n with $x_0 \in W'_n \subset \overline{W'_n} \subset W_n$ to a connected open set $g_n(W'_n)$ in X. Then the connected component of the set $M' \cap \{y'''_n \in g_n(W'_n)\}$ that contains the point y_0 equals the graph

$$\begin{cases} y'''_n &= y'''_n \\ y' &= h(g_n^{-1}(y'''_n)) \\ y''_n &= f_n(g_n^{-1}(y'''_n))\ , \qquad \nu \neq n, \\ y''_\nu &= f_\nu(g_n^{-1}(y'''_n)) \\ y'''_\nu &= g_\nu(g_n^{-1}(y'''_n)) \end{cases}$$

of a holomorphic map $y'''_n \mapsto (y', y''_n, y'''_n, y''_\nu, y'''_\nu)$, $\nu \neq n$, from W'_n to the Banach space $X' \cap \{y'''_n = 0\}$.

Thus M' is a closed split complex Banach submanifold of X' and $\Phi\colon M \to M'$ is a biholomorphism. QED.

The most substantial part of the above proof is to show that holomorphic domination is possible on a separable Banach space. That was done in the reference [P] based upon the work of Lempert in [L]. It might be possible to weaken the axioms (i-iv) perhaps by dropping (iv) and replacing (i), that stands in for holomorphic convexity, by plurisubharmonic domination, i.e., by requiring a continuous plurisubharmonic function $\psi\colon M \to \mathbb{R}$ that dominates the given locally upper bounded function $u\colon M \to \mathbb{R}$. Nevertheless, axioms (i-iv) represent perhaps the ultimate axioms for "Stein Banach manifolds" since any other system for which the desirable Theorem 1 holds must be equivalent with (i-iv). Most known methods of plurisubharmonic domination also yield holomorphic domination, and a 'constructive' procedure for building the functions f_n, g_n in (ii) and (iii) is likely to produce functions that also satisfy (iv). The author doubts whether a successful "Stein theory" could be built up for nonseparable Banach spaces and Banach manifolds. Even for separable Banach manifolds it would be better to restrict attention to the ones modelled on separable Banach spaces with the bounded approximation property (there are virtually no practical separable Banach spaces that do not satisfy the bounded approximation property). If M is a Stein Banach manifold modelled on such a Banach space, then the sheaf cohomology group $H^q(M, S)$ vanishes if $q \geq 1$ and $S \to M$ is a so-called cohesive sheaf defined in [LP] by Lempert et al. The question arises whether M is a Stein Banach manifold if $H^q(M, S) = 0$ for all $q \geq 1$ and all cohesive sheaves $S \to M$. If M is an open subset of a separable Banach space with the bounded approximation property, then the answer is Yes.

References

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